Quotient rings of algebras which are module finite and projective

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Let A be an algebra over the commutative ring R which is finitely generated and projective as an R-module. Then A has a right and left classical ring of quotients.

§ 1. Notation and preliminaries. If a is an element of a ring, then $r(a)$ denotes the right annihilator of a and $l(a)$ the left annihilator of a. We will denote the identity map on a module M by i_M. In this note all rings will have an identity and all modules will be unitary.

We refer the reader to HERSTEIN [2] for a discussion of classical quotient rings, regular elements and the Ore condition. The following lemma can be deduced from BOURBAKI [1], page 88 and exercises page 97.

Lemma 1. *Let F be a finitely generated free module over the commutative ring R and let $\varphi \in \mathrm{End}_R(F)$. The following are equivalent:*

i) $r(\varphi)=0$, ii) $l(\varphi)=0$, iii) $\det \varphi$ *is a regular element of* R.

Also given $\varphi \in \mathrm{End}_R(F)$ there exists $\psi \in \mathrm{End}_R(F)$ such that $\varphi\psi=\psi\varphi=(\det \varphi) i_F$.

§ 2. Generalization of Lemma 1. We extend lemma 1 to projective modules.

Lemma 2. *Let P be a finitely generated projective module over the commutative ring R. If $\alpha \in \mathrm{End}_R(P)$ and $r(\alpha)=0$, then $l(\alpha)=0$ and there exists $\gamma \in \mathrm{End}_R(P)$ such that $\alpha\gamma=\gamma\alpha=c i_P$ where c is a regular element of R. If c is regular in R then $c i_P$ is regular in* $\mathrm{End}_R(P)$.

Proof. Set $S=\mathrm{End}_R(P)$. S is again a finitely generated projective R-module. Given $\alpha \in S$, define $\bar{\alpha} \in \mathrm{End}_R(S)$ by $\bar{\alpha}(\varphi)=\alpha\varphi$ for all $\varphi \in S$. Since S is finitely generated and projective there exists another finitely generated projective R-module S' such that $S \oplus S'=F$, where F is a finitely generated free R-module. Extend $\bar{\alpha}$ to an R-endomorphism $\bar{\alpha}^*$ of F by $\bar{\alpha}^*(s, s')=(\bar{\alpha}(s), s')$. Since $r(\alpha)=0$ we have $\ker(\bar{\alpha})=0$ and so $\ker(\bar{\alpha}^*)=0$. Hence $r(\bar{\alpha}^*)=0$.

Since $\bar{\alpha}^*$ is a right regular element of $\mathrm{End}_R(F)$, by lemma 1 it is also left regular and there exists $\psi \in \mathrm{End}_R(F)$ and c regular in R such that $\bar{\alpha}'\psi = \psi\bar{\alpha}' = ci_F$. It is clear that $\psi|_S$, the restriction of ψ to S, is an element of $\mathrm{End}_R(S)$. Hence we obtain $\alpha\psi|_S = ci_S$. Applying this to $i_P \in S$ we have $\bar{\alpha}(\psi|_S(i_P)) = \alpha(\psi|_S(i_P)) = c \cdot i_P$. Set $\gamma = \psi|_S(i_P) \in S$. Then $\alpha\gamma = c \cdot i_P$. Now $\gamma\alpha = ci_P$ since $\alpha(\gamma\alpha - ci_P) = (\alpha\gamma)\alpha - \alpha(ci_P) = ci_P \cdot \alpha - \alpha \cdot c = 0$ and $r(\alpha) = 0$. It is clear that $c \cdot i_P$ is a regular element of S and so $l(\alpha) = 0$.

§ 3. Theorems. We can now prove

Theorem 3. *Let P be a finitely generated projective module over the commutative ring R. Then $\mathrm{End}_R(P)$ has a classical ring of quotients and this ring can be obtained by inverting regular elements of R.*

Proof. Let $\alpha, \beta \in \mathrm{End}_R(P)$ with α regular. Then we produce γ via lemma 2 such that $\alpha(\gamma\beta) = c \cdot i_P\beta = \beta ci_P$, and $(\beta\gamma)\alpha = \beta(\gamma\alpha) = \beta(ci_P) = ci_P\beta$. Thus $\mathrm{End}_R(P)$ satisfies the right and left Ore condition. It clearly suffices to invert c to obtain this quotient ring.

As a corollary of Theorem 3 we obtain

Theorem 4. *Let A be an algebra over the commutative ring R which is finitely generated and projective as an R-module. Then A has a classical ring of quotients which can be obtained by inverting central regular elements of R.*

Proof. Consider A embedded in $\mathrm{End}_R(A)$ under the map $a \mapsto \bar{a}$ where $\bar{a}(x) = ax$, for all $x \in A$. If a is regular in A then $\bar{a}$ is right regular in $\mathrm{End}_R(A)$. Thus by lemma 2 $\bar{a}$ is also left regular in $\mathrm{End}_R(A)$. To show A satisfies the right and left Ore condition, let $a, b \in A$ with a regular. Then $\bar{a}, \bar{b} \in \mathrm{End}_R(A)$ and $\bar{a}$ is regular in $\mathrm{End}_R(A)$. By theorem 3 we find $\psi \in \mathrm{End}_R(A)$ and c regular in R such that $\bar{a}\psi = \bar{b} \cdot ci_A$. Applying this last equation to $1 \in A$ we get $\bar{a}\psi(1) = \bar{b} \cdot c \cdot 1$. Hence $a\psi(1) = bc$ and A satisfies the right Ore condition. The left Ore condition is similarly verified.

References

[1] N. Bourbaki, *Algèbre*, Chapt. III, 2^{e}ed., Act. Sc. Ind., no. 1044, Hermann (Paris, 1958).
[2] I. N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs 15, Math. Association of America, 1968.

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(Received July 11, 1973)